COMPRESSION OF ELASTIC BODIES UNDER CONDITIONS OF ADHESION (AXISYMMETRIC CASE)

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The problem of compression of elastic bodies was formulated and solved by Hertz in 1882 [1]. Hertz took into account only the first terms in the expansion of the equations of the contact surfaces of the bodies and neglected friction, and therefore he was able to use the apparatus of the theory of Newtonian potential which was developed at that time. In particular, a complete analogy was shown to exist between the contact problem of two elastic bodies and the problem of the gravitational field of a homogeneous ellipsoid.

Numerous publications were devoted during the ensuing years to the application and verification of Hertz's results.

Beliaev [2] calculated the values of the stresses inside the contacting bodies.

Attempts to estimate the influences of friction forces on the contact stresses and on the relative displacements of contacting bodies were made by several authors.

In the papers by Cattaneo [3] and Mindlin [4] the problem of the compression of bodies is solved which have like mechanical characteristics; in this case the pressure distribution, the size and the shape of the contacting surface turn out to be independent of the shear stresses. The problem of the determination of adhesion forces was thus ultimately reduced again to well known problems of potential theory.

The problem of rolling of elastic bodies was considered with the same assumptions in [5].

Principally new problems, from the mathematical point of view, occur in the study of contact problems taking into account friction (or adhesion) in the case when the material properties are not alike.

The first such problem under conditions of a plane state was formulated and solved by Abramov [6].

The problem was formulated thus: to find the stress distribution (normal and shear) at the base of a plane, absolutely rigid die interacting with an elastic half-plane, if its base and the boundary of the semiplane exhibit no slip. Obviously, this is another limiting case: the adhesion forces are so large that they fully inhibit the slippage of the points on the boundary of the semiplane relative to the base of the die.

In this case the boundary conditions of the mixed problem of the semiplane are as follows: outside the contact region the surface tractions are absent (or given), inside the contact region the components of the displacement vector are given.

Later this problem was considered by Galin [7] and Muskhelishvili [8] in a general manner. In another paper [9], Galin considered the problem when the contact area, aside from the region of adhesion, contains also regions of slippage. Friction in these regions was assumed to be linearly related to pressure (Coulomb's law).

The mixed problem of the theory of elasticity for a semi-space with a circular separation for boundary conditions was first formulated in a general manner and solved by the author in paper [10]. By way of examples various cases of interaction of a plane circular die with elastic semispace were considered under conditions of no slip within the complete contact region.

Somewhat later the same problem was investigated by a different method by Ufliand [11].

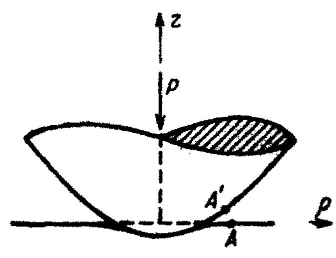
Below the following problem is considered (Fig. 1). We shall assume that, as the compressive force is gradually increased, the contact region increases also gradually. Let it be assumed that for a certain value of the force P the points A_1 and A_2 of the bodies *1* and *2*, respectively, enter into contact. As the compressive force is increased further, these points will be inside the contact region. We shall assume that the adhesion forces will not permit any relative displacement of the two points.

We shall consider the contact of elastic bodies during the process of its development, assuming that the increase in the loading is sufficiently slow, such as to neglect the dynamics of the process. The state of stress will then depend on the loading parameter. The problem is solved under conditions of axial symmetry.

In Section 1 of the paper as an illustration to the solution of the proposed method, the known problem of the indentation of a smooth axi-symmetric die into an elastic half-space is considered. Section 2 contains the solution of the indentation into an elastic half-space of a rigid die under conditions of adhesion (limited case of the formulated problem, when one of the contacting bodies is absolutely rigid).

In Sections 3 and 4 the results of Section 2 are extended to the general case when the elastic constants of the contacting bodies are different. In Section 5 a specific example is considered, which concerns the compression of two rotational paraboloids.

1. Let us consider the problem on the indentation of a smooth, absolutely rigid die into an elastic half-space (Fig. 1). Let $z = \Phi(\rho)$ be the equation of the surface of the die, whereby $\Phi(0) = 0$. As a loading parameter which determines the stage of the state of stress, we take the radius of the contact area a.



The state of stress of the half-space in the absence of shear tractions on the boundary surface may be finally described by means of a single harmonic function $\varphi_3(x, y, z)$, and the stresses σ_z on the boundary and displacements w are expressed by the formulas

$$\sigma_z(x, y, 0) = \frac{E}{2(1-\nu^2)}\varphi_{3z}'(x, y, 0), \qquad w(x, y, 0) = \varphi_3(x, y, 0) \tag{1.1}$$

In the following, taking into account the dependence of φ_3 on the parameter a, we shall designate it by $\varphi_3(\rho, z, a)$.

Let us consider two neighboring states, which correspond to the values of the parameter a and $a + \delta a$. By virtue of the linearity of elasticity problems the difference in the states of stress will also be some state of stress.

Let $f(a)$ be the settlement of the center of the die. Then the boundary conditions for the state of stress considered will be

$$w(\rho, 0, a) = -f(a) - \Phi(\rho) \quad (\rho < a), \qquad \sigma_z(\rho, 0, a) = 0 \quad (\rho > a) \tag{1.2}$$

$$w(\rho, 0, a + \delta a) = -f(a + \delta a) - \Phi(\rho) \qquad (\rho < a + \delta a)$$

$$\sigma_z(\rho, 0, a + \delta a) = 0 \qquad (\rho > a + \delta a)$$

Using (1.1), we obtain

$$\varphi_3(\rho, 0, a) = -f(a) - \Phi(\rho) \quad (\rho < a), \qquad \varphi_{3z}'(\rho, 0, a) = 0 \quad (\rho > a)$$

$$\varphi_3(\rho, 0, a+\delta a) = -f(a+\delta a) - \Phi \quad (\rho < a+\delta a) \tag{1.3}$$

$$\varphi_{3z}'(\rho, 0, a+\delta a) = 0 \quad (\rho > a+\delta a)$$

From (1.3), passing to the limit as $\delta a \to 0$, we have

$$\frac{\partial \varphi_3(\rho, 0, a)}{\partial a} = -f'(a) \quad (\rho < a), \qquad \frac{\partial \varphi_{3z}'(\rho, 0, a)}{\partial a} = 0 \quad (\rho > a) \tag{1.4}$$

The derivative of a harmonic function is also a harmonic function.

Conditions (1.4) are sufficient for the determination of a harmonic function $\varphi_{3a}(\rho, z, a)$. Obviously, this function will be the solution of the problem of the indentation into an elastic half-space of a circular die with a smooth plane base by a depth $f'(a)$. Thus, the expression is valid

$$\frac{\partial \varphi_3(\rho, z, a)}{\partial a} = f'(a)\,\varphi_0(\rho, z, a) \tag{1.5}$$

where $\varphi_0(\rho, z, a)$ is a known function, corresponding to the solution of the problem on the indentation into an elastic half-space of a circular die by unit depth. From (1.5) it follows

$$\varphi_3(\rho, z, a) = \int_0^a f'(t)\,\varphi_0(\rho, z, t)\,dt \tag{1.6}$$

Assuming $z = 0$ in equation (1.6) and taking into account condition (1.2), we obtain an integral equation for the determination of the unknown function $f(a)$

$$-\Phi(\rho) = \int_0^a f'(t)\,[\varphi_0(\rho, 0, t) + 1]\,dt \tag{1.7}$$

On the boundary of the half-space $\varphi_0(\rho, z, t)$ takes on the values

$$\varphi_0(\rho, 0, t) = -1 \quad (t > \rho), \qquad \varphi_0(\rho, 0, t) = -\frac{2}{\pi}\sin^{-1}\frac{t}{\rho} \quad (t < \rho)$$

Thus, equation (1.7) takes on the form

$$-\Phi(\rho) = \int_0^\rho f'(t)\left[-\frac{2}{\pi}\sin^{-1}\frac{t}{\rho} + 1\right]dt \tag{1.8}$$

Differentiating (1.8) with respect to ρ, we obtain

$$-\Phi'(\rho)=\frac{2}{\pi}\int_0^\rho f'(t)\frac{t\,dt}{\rho\sqrt{\rho^2-t^2}} \qquad (1.9)$$

This is Abel's equation. Its solution is of the form

$$f'(t)=-\frac{1}{\pi}\int_0^t (t^2-r^2)^{-\frac{1}{2}}\left[\Phi'(\rho)+r\Phi''(r)\right]dr$$

and the following is valid

$$\varphi_{0z}'(\rho,0,t)=\frac{2}{\pi\sqrt{t^2-\rho^2}} \quad (\rho<t), \qquad \varphi_{0z}'(\rho,0,t)=0 \quad (\rho>t) \qquad (1.10)$$

Using (1.6) we find

$$\varphi_z'(\rho,0,a)=-\frac{1}{2\pi}\int_\rho^a\left\{\frac{1}{t}\frac{d}{dt}\int_0^t\frac{\Phi'(r)\,r\,dt}{\sqrt{t^2-r^2}}\right\}\frac{dt}{\sqrt{t^2-r^2}} \qquad (1.11)$$

After simple transformations, using (1.1), we obtain the known formula for the pressure under a die in the case when the pressures on its boundary are bounded [12]

$$\sigma_z(\rho,0,a)=-\frac{E}{\pi(1-\nu^2)}\int_\rho^a\frac{dt}{\sqrt{t^2-\rho^2}}\int_0^t\frac{\Phi'(r)+\Phi''(r)\,r}{\sqrt{t^2-r^2}}\,dr \qquad (1.12)$$

For the following we note that if $f'(t)=t^m$, then $\Phi(\rho)=C_m\rho^m$.

To find the value of the constant C_m we use the known formula

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad (1.13)$$

and after transformations

$$\int_0^1 x^{2\alpha-1}(1-x^2)^{\beta-1}\,dx=\frac{1}{2}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(a+\beta)}$$

In the case considered $\beta=1/2$, $\alpha=1/2\,m+1$. Thus, with the aid of the substitution $f'(t)=t^m$ from (1.9) we obtain

$$C_m=-\frac{2}{\pi}\frac{\Gamma(1/2)\,\Gamma(1/2\,m+2)}{\Gamma(1/2\,m+3)} \qquad (1.14)$$

In the general case, when $\Phi'(\rho)$ is a polynomial

$$\Phi'(\rho)=\sum_{n=0}^{k}A_m\rho^m$$

the function $f'(t)$ is also a polynomial

$$f'(t) = \sum_{m=0}^{k} \frac{A_m}{C_m} t^m \tag{1.15}$$

2. Let us consider now the indentation of an absolutely rigid die, with axial symmetry, into an elastic half-space. We shall assume that the adhesion between the surfaces of the die and the half-space is so large that slippage on the contact area is completely absent. The boundary points of the elastic half-space, which enter into a contact with the surface of the die, adhere to it and move together with the die, that is, parallel to the z-axis, during the further evolution of the process. Mathematically, this condition is expressed by the function of radial displacements $u_z(\rho, z, a)$ in the contact area being independent of the radius of the die a.

Thus, the boundary conditions of the problem may be written in the form

$$\begin{gathered} w(\rho, 0, a) = -f(a) - \Phi(\rho), \quad u_r(\rho, 0, a) = F(\rho) \quad (\rho < a) \\ \sigma_z(\rho, 0, a) = 0, \quad \tau_{zr}(\rho, 0, a) = 0 \quad (\rho > a) \end{gathered} \tag{2.1}$$

Using the general solution of the equations of the theory of elasticity in the form of Trefftz, the state of stress for the half-space may be described with the aid of two harmonic functions. In the case of axial symmetry the relations (2.1) take on the form

$$\begin{gathered} \varphi_3(\rho, 0, a) = -f(a) - \Phi(\rho), \qquad \varphi_{4z}(\rho, 0, a) = \frac{\partial F(\rho)}{\partial \rho} + \frac{F(\rho)}{\rho} \quad (\rho < a) \\ \varphi_{3z}(\rho, 0, a) - \frac{1}{A}\varphi_{4z}'(\rho, 0, a) = 0, \quad \varphi_3(\rho, 0, a) - A\varphi_4(\rho, 0, a) = 0 \quad (a < \rho) \end{gathered} \tag{2.2}$$

Here

$$A = \frac{2\mu + \lambda}{\mu} = 2\,\frac{1-\nu}{2-2\nu}$$

For a neighboring state with a radius of the area $a + \delta a$, the relations (2.2) take on the form

$$\begin{gathered} \varphi_3(\rho, 0, a + \delta a) = -f(a + \dot{\delta} a) - \Phi(\rho) \\ \varphi_{4z}'(\rho, 0, a + \delta a) = \frac{\partial F(\rho)}{\partial \rho} + \frac{F(\rho)}{\partial \rho} \quad (\rho < a + \delta a) \\ \varphi_{3z}'(\rho, 0, a + \delta a) - \frac{1}{A}\varphi_{4z}'(\rho, 0, a + \delta a) = 0 \\ \varphi_3(\rho, 0, a + \delta a) - A\varphi_4(\rho, 0, a + \delta a) = 0 \quad (a + \delta a < \rho) \end{gathered} \tag{2.3}$$

Just as in Section 1, we consider the difference of the state of stress as $\delta a \to 0$; then we obtain

$$\frac{\partial\varphi_3(\rho, 0, a)}{\partial a} = -f'(a), \qquad \frac{\partial\varphi_{4z}'(\rho, 0, a)}{\partial a} = 0 \qquad (\rho < a)$$

$$\frac{\partial\varphi_{3z}'(\rho, 0, a)}{\partial a} - \frac{1}{A}\frac{\partial\varphi_{4z}'(\rho, 0, a)}{\partial a} = 0, \quad \frac{\partial\varphi_3(\rho, 0, a)}{\partial a} - A\frac{\partial\varphi_4(\rho, 0, a)}{\partial a} = 0 \quad (a < \rho) \tag{2.4}$$

Conditions (3.4) indicate that the state of stress in a half-space, described by the harmonic functions $\partial\varphi_3(\rho,\ z,\ a)/\partial a$, $\partial\varphi_4(\rho,\ t,\ a)/\partial a$, corresponds to the symmetric indentation of a plane circular die into an elastic half-space under the conditions of adhesion [no slip] by a depth $f'(a)$. Thus, the sought functions can be represented in the form

$$\frac{\partial\varphi_3(\rho, z, a)}{\partial a} = f'(a)\,\varphi_{30}(\rho,\ z,\ n), \qquad \frac{\partial\varphi_4(\rho, z, a)}{\partial a} = f'(a)\,\varphi_{40}(\rho,\ z,\ a) \tag{2.5}$$

where φ_{30}, φ_{40} are known functions [10], which correspond to the solution of the problem of the indentation into an elastic half-space of a plane circular die by unit depth under conditions of adhesion

$$\varphi_3(\rho,\ z,\ a) = \int_0^a f'(t)\,\varphi_{30}(\rho,\ z,\ t)\,dt \tag{2.6}$$

$$\varphi_4(\rho,\ z,\ a) = \int_0^a f'(t)\,\varphi_{40}(\rho,\ z,\ t)\,dt \tag{2.7}$$

Setting $z = 0$ in equation (2.6) and considering (2.2), we obtain the integral equation

$$-\Phi(\rho) = \int_0^a f'(t)\,[\varphi_{30}(\rho, 0, t) + 1]\,dt \tag{2.8}$$

On the boundary of half-space the function $\varphi_{30}(p,\ 0,\ t)$ takes on the values

$$\varphi_{30}(\rho, 0, t) = \frac{1}{2\pi}\int_0^\rho u_{1x}'(x, 0, t)\frac{dx}{\sqrt{\rho^2 - x^2}} \tag{2.9}$$

where

$$u_{1x}'(x,\ 0,\ t) = -4 \quad (x < t), \qquad u_{1x}'(x,\ 0,\ t) = \left[-4 + 4\cos\theta\ln\frac{x-t}{x+t}\right] \quad (x > t)$$

$$\theta = \frac{1}{2\pi}\ln\frac{A+1}{A-1} = \frac{1}{2\pi}\ln(3 - 4\nu) \tag{2.10}$$

Thus

$$\varphi_{30}(\rho, 0, t) = -1 \qquad (\rho < t)$$

$$\varphi_{30}(\rho, 0, t) = -1 + \frac{2}{\pi}\int_0^\rho \cos\left(\theta \ln\frac{x-t}{x+t}\right)\frac{dt}{\sqrt{\rho^2 - x^2}} \qquad (\rho > t) \qquad (2.11)$$

Thus, the equation for the determination of the sought function $f'(t)$ takes on the form

$$-\Phi(\rho) = \frac{2}{\pi}\int_t^\rho f'(t)\, dt \int_t^\rho \frac{\chi(x,t)\, dx}{\sqrt{\rho^2 - x^2}}, \quad \chi(x,t) = \cos\left(\theta \ln\frac{x-t}{x+t}\right) \qquad (2.12)$$

Changing the order of integration in (2.12), we obtain

$$-\Phi(\rho) = \frac{2}{\pi}\int_0^\rho \frac{dx}{\sqrt{\rho^2 - x^2}}\int_0^x f'(t) \cos\theta \ln\frac{x-t}{x+t}\, dt \qquad (2.13)$$

Now it becomes obvious that the discovered property of equation (1.9) is valid also for equation (2.13). Substituting $f'(t) = t^m$, we find that $\Phi(\rho) = d_m \rho^{m+1}$, where

$$d_m = -\frac{2}{\pi}\int_0^1 \frac{x^{m+1}}{\sqrt{1 - x^2}}\, dx \int_0^1 t^m \cos\left(\theta \ln\frac{1-t}{1+t}\right) dt \qquad (2.14)$$

Consequently, equation (2.13) may always be inverted whenever $\Phi(\rho)$ is a polynomial.

The pressure at the base of the plane circular die with unit settlement is determined from the formula

$$\sigma_{0z}(\rho, 0, t) = \frac{8\theta t\mu(2\mu + \lambda)\sqrt{A^2 - 1}}{\pi(3\mu + \lambda)}\int_0^\rho \frac{\chi(x,t)dx}{\sqrt{\rho^2 - x^2}(t^2 - x^2)} \qquad (\rho < t)$$

$$\sigma_{0z}(\rho, 0, t) = 0 \qquad (\rho > t) \qquad (2.15)$$

Thus, the pressure in the general case of indentation of an absolutely rigid die with axial symmetry under the conditions of adhesion [no slippage] is determined by the expression

$$\sigma_z(\rho, 0, a) = \frac{8\theta\mu(2\mu + \lambda)\sqrt{A^2 - 1}}{\pi(3\mu + \lambda)}\int_\rho^a f'(t)\, t dt \int_0^\rho \frac{\chi(x, t)dt}{\sqrt{\rho^2 - x^2}(t^2 - x^2)} \qquad (2.16)$$

3. Let us consider the problem of compression of two bodies possessing axial symmetry. As is usual we shall assume that the contact area is small as compared to the linear dimensions of the contacting bodies and we shall replace them by two half-spaces. In the sequel all quantities pertaining to the "lower" half-space ($z \leqslant 0$), will be indicated by subscript 1; quantities pertaining to the "upper" half-space, by subscript 2.

As before, the radius of the contact area a will be taken as the parameter which determines various states of stress.

Let the equations of the surfaces of undeformed half-spaces be, respectively

$$z = \Phi_1(\rho), \qquad z = \Phi_2(\rho) \tag{3.1}$$

The boundary conditions of the problem may be written in the form

$$\begin{gathered}
\sigma_{1z}(\rho, 0, a) = \sigma_{2z}(\rho, 0, a), \qquad \tau_{1z\rho}(\rho, 0, a) = \tau_{2z\rho}(\rho, 0, a) \qquad (0 < \rho < \infty) \\
w_1(\rho, 0, a) - w_2(\rho, 0, a) = \Phi_1(\rho) - \Phi_2(\rho) - f(a) \\
u_{1r}(\rho, 0, a) - u_{2r}(\rho, 0, a) = F(\rho) \qquad (\rho < a) \\
\sigma_{1z}(\rho, 0, a) = 0, \qquad \tau_{1z\rho}(\rho, 0, a) = 0 \qquad (\rho > a)
\end{gathered} \tag{3.2}$$

Comparing, as before, two infinitely close states and considering that $F(\rho)$ does not depend on a, we find

$$\begin{gathered}
\frac{\partial w_1(\rho, 0, a)}{\partial a} - \frac{\partial w_2(\rho, 0, a)}{\partial a} = -f'(a) \\
\frac{\partial u_{1r}(\rho, 0, a)}{\partial a} - \frac{\partial u_{2r}(\rho, 0, a)}{\partial a} = 0 \qquad (\rho < a) \\
\frac{\partial \sigma_{1z}(\rho, 0, a)}{\partial a} = 0, \qquad \frac{\partial \tau_{1rz}(\rho, 0, a)}{\partial a} = 0 \qquad (a < \rho) \\
\frac{\partial \sigma_{1z}(\rho, 0, a)}{\partial a} = \frac{\partial \sigma_{2z}(\rho, 0, a)}{\partial a}, \qquad \frac{\partial \tau_{1rz}(\rho, 0, a)}{\partial a} = \frac{\partial \tau_{2rz}(\rho, 0, a)}{\partial a} \qquad (0 < \rho < \infty)
\end{gathered} \tag{3.3}$$

The boundary conditions (3.3) determine, within the accuracy of the factor $f'(a)$, the following problem: two half-spaces are bonded on a circular area and then are displaced one with respect to the other along the normal to the boundary by a unit amount. If the quantities which determine the solution to this problem are given the indices 10 and 20 for the lower and the upper semi-space, respectively, then the following relations are valid

$$\begin{gathered}
\frac{\partial w_1(\rho, z, a)}{\partial a} = -f'(a)\, w_{10}(\rho, z, a), \qquad \frac{\partial w_2(\rho, z, a)}{\partial a} = -f'(a)\, w_{20}(\rho, z, a) \\
\frac{\partial \sigma_{1z}(\rho, 0, a)}{\partial a} = -f'(a)\, \sigma_{10z}(\rho, z, a) \qquad \text{etc.}
\end{gathered} \tag{3.4}$$

or, after integration with respect to a

$$
\begin{aligned}
w_1(\rho, z, a) &= -\int_0^a f'(t)\ w_{10}(\rho, z, t)\, dt \\
w_2(\rho, z, a) &= -\int_0^a f'(t)\ w_{20}(\rho, z, t)\, dt \\
\sigma_{1z}(\rho, z, a) &= -\int_0^a f'(t)\ \sigma_{10z}(\rho, z, t)\, dt \qquad \text{etc.}
\end{aligned} \tag{3.5}
$$

The first two equations (3.5), together with (3.2), yield an integral equation for the determination of the sought function

$$
\Phi_1(\rho) - \Phi_2(\rho) = -\int_0^a f'(t)\, [w_{10}(\rho, 0, t) - w_{20}(\rho, 0, t) - 1]\, dt \tag{3.6}
$$

4. Here we solve the auxiliary mixed problem, whose special case was formulated in Section 3: to determine the state of stress and deformation of two half-spaces, occupying, respectively, the regions $z \leqslant 0$ (lower or first half-space) and $z \geqslant 0$ (upper or second half-space), if the conditions are prescribed

$$
\begin{gathered}
u_1(x, y, 0) - u_2(x, y, 0) = u(x, y) \\
v_1(x, y, 0) - v_2(x, y, 0) = v(x, y) \quad \text{in region } S \\
w\ (x, y, 0) - w_2(x, y, 0) = w(x, y) \\
\sigma_{1z}(x, y, 0) = \sigma_{2z}(x, y, 0), \qquad \tau_{1zx}(x, y, 0) = \tau_{2zx}\ x, y, 0) \\
\tau_{1zy}(x, y, 0) = \tau_{2zy}(x, y, 0)
\end{gathered} \tag{4.1}
$$

where $u(x, u)$, $v(x, y)$, $w(x, y)$ are known functions.

Further, we shall assume that outside the region S the surfaces are not subjected to any loads and that the stresses and displacements in both half-spaces are vanishing at infinity. Let it be assumed that on a portion of the boundary S of the elastic half-space $z \leqslant 0$ the loads are applied

$$
\sigma_{1z}(x, y, 0) = N(x, y), \quad \tau_{1zx}(x, y, 0) = L(x, y), \quad \tau_{1zy}(x, y, 0) = M(x, y) \tag{4.2}
$$

The displacements of the boundary points are determined by formulas [3]

$$
\begin{aligned}
u_1(x, y, 0) = \frac{1}{2\pi\mu_1}\int_S \frac{L}{r}\, dS - \frac{\mu}{4\pi\mu_1(\lambda_1+\mu_1)}\int_S L\frac{\partial^2 r}{\partial x^2}\, ds - \\
- \frac{\lambda_1}{4\pi\mu_1(\lambda_1+\mu_1)}\int_S M\frac{\partial^2 r}{\partial x \partial y}\, ds - \frac{1}{4\pi(\lambda_1+\mu_1)}\int_S N\frac{\partial}{\partial x}\ln r\, ds
\end{aligned} \tag{4.3}
$$

$$v_1(x,y,0)=\frac{1}{2\pi\mu_1}\int\limits_S\frac{M}{r}ds-\frac{\lambda_1}{4\pi\mu_1(\lambda_1+\mu_1)}\int\limits_S M\frac{\partial^2 r}{\partial y^2}ds-$$

$$-\frac{\lambda_1}{4\pi\mu_1(\lambda_1+\mu_1)}\int L\frac{\partial^2 r}{\partial x\partial y}-\frac{1}{4\pi(\lambda_1+\mu_1)}\int\limits_S N\frac{\partial}{\partial y}\ln r\ ds$$

$$w_1(x,y,0)=\frac{\lambda_1+2\mu_1}{4\pi\mu(\lambda_1+\mu_1)}\int\limits_S\frac{N}{r}ds+\frac{1}{4\pi(\lambda_1+\mu_1)}\int\limits_S\left(L\frac{\partial\ln r}{\partial x}+M\frac{\partial\ln r}{\partial y}\right)ds$$

Here and in the following

$$r=\sqrt{(x-x_1)^2+(y-y_1)^2}\qquad\left(\Lambda_i=\frac{\lambda i+2\mu i}{\mu_i(\lambda_i+\mu_i)},\quad i=1,2\right)\tag{4.4}$$

Formulas (4.3) are transformed to the form

$$\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}=\frac{\Lambda_1}{4\pi}\int\limits_S\left(\frac{\partial L}{\partial x}+\frac{\partial M}{\partial y}\right)\frac{\partial s}{r}-\frac{1}{4\pi(\lambda_1+\mu_1)}\int\limits_S\frac{N}{r^2}ds\tag{4.5}$$

$$w_1=\frac{\Lambda_1}{4\pi}\int\limits_S\frac{N}{r}-\frac{1}{4\pi(\lambda_1+\mu_1)}\int\limits_S\left(\frac{\partial L}{\partial x}+\frac{\partial M}{\partial y}\right)\ln r ds$$

Analogous formulas are valid for the upper half-space with the difference that the sign of the first integral has to be changed.

Thus, we obtain

$$\frac{\partial}{\partial x}(u_1-u_2)+\frac{\partial}{\partial y}(v_1-v_2)=\frac{\Lambda_1+\Lambda_2}{4\pi}\int\limits_S\left(\frac{\partial L}{\partial x}+\frac{\partial M}{\partial y}\right)\frac{ds}{r}-$$

$$-\frac{1}{4\pi}\left[\frac{1}{\lambda_1+\mu_1}-\frac{1}{\lambda_2+\mu_2}\right]\int\limits_S\frac{N}{r^2}ds\tag{4.6}$$

$$w_1-w_2=\frac{\Lambda_1+\Lambda_2}{4\pi}\int\limits_S\frac{N}{r}-\frac{1}{4\pi}\left[\frac{1}{\lambda_1+\mu_1}-\frac{1}{\lambda_2+\mu_2}\right]\int\limits_S\left(\frac{\partial L}{\partial x}+\frac{\partial M}{\partial y}\right)\ln r ds$$

In the case of axial symmetry considered, the quantities entering into (4.6) determine completely the state of stress and deformation.

Expression (4.6) may be considered as an integral equation for the determination of the quantities N, $\partial L/\partial x+\partial M/\partial y$. On the other hand, when these quantities are found, using formulas (4.6) one can find the values of the functions w_1-w_2 outside S.

For the elastic half-space $z\leqslant 0$ with elastic constants λ, μ, which are determined by the formulas

$$\frac{1}{\lambda+\mu}=\frac{1}{\lambda_1+\mu_1}-\frac{1}{\lambda_2+\mu_2}, \qquad \frac{\lambda+2\mu}{\mu(\lambda+\mu)}=\frac{\lambda_1+2\mu_1}{\mu_1(\lambda_1+\mu_1)}+\frac{\lambda_2+2\mu_2}{\mu_2(\lambda_2+\mu_2)} \quad (4.7)$$

the integral equations of the mixed problem are equivalent to equations (4.6). Thereby the force and geometric factors of both problems coincide on the boundary surface.

Consequently, the solution of the problem in Section 3 is automatically extended to the general case of compression of elastic bodies, with the only difference that in place of elastic constants λ and μ we have to substitute their values from (4.7) into the formulas.

5. As an example we consider the indentation of an absolutely rigid sphere into an elastic half-plane under the conditions of adhesion [no slip].

Limiting ourselves in the expansion of the equation of the sphere of radius R by the first terms, as is usually done in such types of problems, we obtain

$$\Phi(\rho)=\frac{1}{2R^{\circ}}\rho^2 \quad (5.1)$$

In accordance with (2.14) we obtain

$$f'(t)=\frac{1}{Rd_1}\rho \quad (5.2)$$

We indicate below the values of the quantities d_1 for various values of Poisson's ratio ν

$\nu=0$	0.1	0.2	0.3	0.4	0.5
$d_1=-0.225$	-0.234	-0.240	-0.244	-0.248	-0.250

The total loading P, which produces a given settlement of the die, using (2.16), is found to be

$$P=\frac{8\theta\mu(2\mu+\lambda)\sqrt{A^2-1}}{\pi(3\mu+\lambda)}\int_0^a 2\pi\,\rho d\rho\int_\rho^a f'(t)\,t\,dt\int_0^\rho\frac{\chi(x,t)\,dx}{\sqrt{\rho^2-x^2}\,(t^2-x^2)} \quad (5.3)$$

Substituting the expression (5.2) for $f'(t)$, and interchanging the order of integration, we obtain

$$P=\frac{16\theta\mu(2\mu+\lambda)\sqrt{A^2-1}}{(3\mu+\lambda)Rd_1}\int_0^a dx\int_x^a\frac{t^2}{\sqrt{t^2-x^2}}\,\chi(x,t) \quad (5.4)$$

Changing the order of integration, we have

$$P = \frac{16\theta\mu(2\mu+\lambda)\sqrt{A^2-1}}{(3\mu+\lambda)Rd_1}\int_0^a t^2 dt \int_0^t \frac{\chi(x,t)}{\sqrt{t^2-x^2}} \tag{5.5}$$

The inner integral is easily evaluated by the method of Muskhelishvili and is equal to

$$\frac{\pi}{2}\frac{\sqrt{A+1}}{A}$$

Finally, the following expression for the total force P is obtained

$$P = \frac{8\pi\theta\mu(2\mu+\lambda)(A^2-1)a^3}{3(3\mu+\lambda)Rd_1 A} \tag{5.6}$$

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